

ON SUPER-JORDANIAN $\mathcal{U}_h(sl(N|1))$ ALGEBRA

In memory of our friend Professor Daniel Arnaudon

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Abstract

A nonlinear realization of the nonstandard (super-Jordanian) version of $\mathcal{U}(sl(N|1))$ is given, for all N .

1 Introduction

Jordanian and super-Jordanian quantum algebras have been recently used for several applications in physical problems. For instance, super-Jordanian $\mathcal{U}_h(osp(1|2))$ algebra has been understood as the κ -deformation of the symmetry algebra of the super-conformal mechanics [1]. In another context, integrable deformed Hamiltonian systems have been introduced [2] via Poisson coalgebra associated with quantized Jordanian $\mathcal{U}_h(sl(2))$ algebra. We believe that fully developed Hopf coalgebraic structure in a deformed basis for the $\mathcal{U}_h(sl(N|1))$ presented here will be useful in building and studying similar deformed fermionic integrable models. Furthermore, using the corepresentation structure of the function algebra dually related to the universal enveloping algebra, a general method of constructing noncommutative (super)spaces has been recently developed [3] in the context of the quantum supergroup $OSp_q(1|2)$. Application of this method to the case of the dual quantum supergroup $SL_h(N|1)$ will lead to new quantum superspaces inherently containing a *dimensional* deformation parameter. Influenced by these observations here we introduce the super-Jordanian $\mathcal{U}_h(sl(N|1))$ algebra in a deformed basis set.

In a series of papers [4, 5, 6, 7], we have proposed a new scheme which permits the construction of the nonstandard version $\mathcal{U}_h(\mathfrak{g})$ of an enveloping (super)algebra $\mathcal{U}(\mathfrak{g})$ by a suitable contraction, from the corresponding standard ones $\mathcal{U}_q(\mathfrak{g})$. Our method hinges on obtaining the \mathcal{R}_h -matrix, for all dimensions, of a (super)Jordanian quantum (super)algebra $\mathcal{U}_h(\mathfrak{g})$ from the \mathcal{R}_q -matrix associated to the standard quantum (super)algebra $\mathcal{U}_q(\mathfrak{g})$ through a specific transformation G (singular in the $q \rightarrow 1$ limit), as follows:

$$\mathcal{R}_h = \lim_{q \rightarrow 1} \left[G^{-1} \otimes G^{-1} \right] \mathcal{R}_q [G \otimes G], \quad (1.1)$$

where, for example, $G = E_q \left(\frac{\hbar \hat{e}_{1N}}{q-1} \right)$ for $\mathcal{U}_q(sl(N))$ (\hat{e}_{1N} is the longest positive root generator of $\mathcal{U}_q(sl(N))$) and $G = E_{q^2} \left(\frac{\hbar \hat{e}^2}{q^2-1} \right)$ for $\mathcal{U}_q(osp(1|2))$ (\hat{e} is the fermionic positive simple root generator of $\mathcal{U}_q(osp(1|2))$). The deformed exponential map E_q is defined by

$$E_q(\eta) = \sum_{n=0}^{\infty} \frac{(\eta)^n}{[n]_q!}, \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! = [n]_q \times [n-1]_q!, \quad [0]_q! = 1. \quad (1.2)$$

For the transformed matrix, the singularities, however, cancel yielding a well-defined construction. This procedure yields a nonstandard deformation along with a nonlinear map of the \hbar -Borel subalgebra on the corresponding classical Borel subalgebra, which can be artfully extended to the whole (super)algebra. The Jordanian quantum algebra $\mathcal{U}_h(sl(N))$ arising from the process cited above corresponds to the classical matrix $r = h_{1N} \wedge e_{1N}$. Therefore, the universal \mathcal{R}_h -matrix of the full $\mathcal{U}_h(sl(N))$ Hopf algebra, obtained, coincides with the universal \mathcal{R}_h -matrix of the $\mathcal{U}_h(sl(2))$ Hopf subalgebra [19] associated with the highest roots. In the case of $\mathcal{U}(osp(1|2))^*$, the super-Jordanian quantum super-algebra $\mathcal{U}_h(osp(1|2))$ occurred from our treatment is associated to the classical matrix $r = h \wedge e^2 - e \wedge e$. The advantages of our technic are: (1) With an

*The recent work shows that there exist three distinct bialgebra structure on $osp(1|2)$ and all of them are coboundary. We therefore have three distinct quantization of $osp(1|2)$.

appropriate choice of basis, the Jordanian quantum Hopf (super)algebra, obtained by our process, can be endowed with a relatively simpler coalgebraic structure; (2) Our nonlinear map permits immediate explicit construction of the finite-dimensional irreducible representations.

Let us just mention that in general, nonstandard quantum algebras are obtained by applying Drinfeld twist [8] to the corresponding Lie algebras (see [9, 10, 11, 12, 13, 14, 15] and refs. there in. The twist deformation of super-algebras was also discussed in the litterature: [16, 1] ($\mathcal{U}(\mathfrak{osp}(1|2))$ case), [17] ($\mathcal{U}(\mathfrak{osp}(1|4))$ case) and [18] (general super-algebra case). We will not consider this way here.

The main object of this paper is to present how our contraction procedure work for $\mathcal{U}(\mathfrak{sl}(N|1))$ super-algebra for obtaining the nonstandard version $\mathcal{U}_h(\mathfrak{sl}(N|1))$. For simplicity, we will limit here ourselves to $\mathcal{U}(\mathfrak{sl}(2|1))$ and $\mathcal{U}(\mathfrak{sl}(3|1))$. The construction of higher dimensional super-algebras $\mathcal{U}_h(\mathfrak{sl}(N|1))$ is presented, briefly, in the end of this paper. The manuscript is organized as follows: the super-Jordanian quantum super-algebra $\mathcal{U}_h(\mathfrak{sl}(2|1))$ is introduced via a nonlinear map and proved to be a Hopf Algebra. Higher dimensional super-algebras $\mathcal{U}_h(\mathfrak{sl}(N|1))$, $N \geq 3$, are presented in sections 3 and 4. We conclude in section 5.

2 $\mathcal{U}_h(\mathfrak{sl}(2|1))$: contraction, nonlinear map and Hopf structure

Let us just recall the more important points concerning $\mathfrak{sl}(2|1)$: Let $A = (a_{ij})$ be the 2×2 matrix given by $a_{11} = 2$, $a_{12} = a_{21} = -1$ and $a_{22} = 0$. The Lie Hopf superalgebra $\mathcal{U}(\mathfrak{sl}(2|1))$ is generated by the generators h_i , e_i and f_i , $i = 1, 2$, where h_1 , h_2 , e_1 and f_1 are even ($\deg(h_1) = \deg(h_2) = \deg(e_1) = \deg(f_1) = 0$), while e_2 and f_2 , are odd ($\deg(e_2) = \deg(f_2) = 1$), and the commutation relations

$$\begin{aligned} [h_i, h_j] &= 0, & [h_i, e_j] &= a_{ij}e_j, & [h_i, f_j] &= -a_{ij}f_j, & [e_i, f_j] &= \delta_{ij}h_i, \\ [e_2, e_2] &= [f_2, f_2] = 0, & [e_1, [e_1, e_2]] &= [f_1, [f_1, f_2]] = 0. \end{aligned} \quad (2.1)$$

The two last equations are called the Serre relations. The commutator $[\ , \]$ is understood as the \mathbb{Z}_2 -graded one: $[a, b] = ab - (-)^{\deg(a)\deg(b)}ba$. Defining

$$e_3 = e_1e_2 - e_2e_1, \quad f_3 = f_2f_1 - f_1f_2, \quad (2.2)$$

we obtain

$$\begin{aligned} [e_1, e_3] &= 0, & [f_3, f_1] &= 0, & [e_2, e_3] &= 0, & [f_2, f_3] &= 0, \\ e_3^2 &= f_3^2 = 0, & [e_3, f_3] &= h_1 + h_2 \equiv h_3, & [f_1, e_3] &= e_2, & \text{etc.} \end{aligned} \quad (2.3)$$

Let us just mention that there is a \mathbb{C} -algebra automorphism ϕ of $\mathcal{U}(\mathfrak{sl}(2|1))$ such that

$$\phi : (h_1, h_2, h_3, e_1, e_2, e_3, f_1, f_2, f_3) \rightarrow (h_1, -h_3, -h_2, e_1, f_3, -f_2, f_1, -e_3, e_2). \quad (2.4)$$

The quasitriangular quantum Hopf superalgebra $\mathcal{U}_q(\mathfrak{sl}(2|1))$ (q is an arbitrary complex number), by analogy with $\mathcal{U}(\mathfrak{sl}(2|1))$, is generated by six elements \hat{h}_i , \hat{e}_i and \hat{f}_i , $i = 1, 2$, under the relations

$$\begin{aligned} [\hat{h}_i, \hat{h}_j] &= 0, & [\hat{h}_i, \hat{e}_j] &= a_{ij}\hat{e}_j, & [\hat{h}_i, \hat{f}_j] &= -a_{ij}\hat{f}_j, \\ [\hat{e}_i, \hat{f}_j] &= \delta_{ij} \frac{q^{\hat{h}_i} - q^{-\hat{h}_i}}{q - q^{-1}}, & \hat{e}_2^2 &= \hat{f}_2^2 = 0, \\ \hat{e}_1^2\hat{e}_2 - (q + q^{-1})\hat{e}_1\hat{e}_2\hat{e}_1 + \hat{e}_2\hat{e}_1^2 &= \hat{f}_1^2\hat{f}_2 - (q + q^{-1})\hat{f}_1\hat{f}_2\hat{f}_1 + \hat{f}_2\hat{f}_1^2 = 0. \end{aligned} \quad (2.5)$$

All generators are even except for \hat{e}_2 and \hat{f}_2 which are odd. The coproducts, counits and antipodes are given by

$$\begin{aligned} \Delta(\hat{e}_i) &= \hat{e}_i \otimes q^{\hat{h}_i/2} + q^{-\hat{h}_i/2} \otimes \hat{e}_i, & \epsilon(\hat{e}_i) &= 0, & S(\hat{e}_i) &= -q^{\hat{h}_i/2}\hat{e}_iq^{-\hat{h}_i/2}, \\ \Delta(\hat{f}_i) &= \hat{f}_i \otimes q^{\hat{h}_i/2} + q^{-\hat{h}_i/2} \otimes \hat{f}_i, & \epsilon(\hat{f}_i) &= 0, & S(\hat{f}_i) &= -q^{\hat{h}_i/2}\hat{f}_iq^{-\hat{h}_i/2}, \\ \Delta(\hat{h}_i) &= \hat{h}_i \otimes 1 + 1 \otimes \hat{h}_i, & \epsilon(\hat{h}_i) &= 0, & S(\hat{h}_i) &= -\hat{h}_i, \end{aligned} \quad (2.6)$$

The universal \mathcal{R} -matrix is given in refs. [20, 21]. Note that the definition of the Hopf superalgebra differs from that of the usual Hopf algebra by the supercommutativity of tensor product, i.e. $(a \otimes b)(c \otimes d) = (-1)^{\deg(b)\deg(c)}(ac \otimes bd)$. For later use, we note that the fundamental representation of (2.5) is spanned by

$$\begin{aligned} \hat{h}_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \hat{e}_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \hat{f}_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \hat{h}_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \hat{e}_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, & \hat{f}_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned} \quad (2.7)$$

2.1 Contraction Process

Following [4], the \mathcal{R}_h (h is an arbitrary complex number) matrix of the super-Jordanian quantum superalgebra $\mathcal{U}_h(sl(2|1))$, for arbitrary representations in the two tensor product sectors, can be also obtained from the \mathcal{R}_q -matrix associated with the Drinfeld-Jimbo quantum superalgebra $\mathcal{U}_q(sl(2|1))$ through a specific contraction. For simplicity and brevity, let us start with (fundamental irrep.) \otimes (fundamental irrep.). The \mathcal{R}_q -matrix of $\mathcal{U}_q(sl(2|1))$ superalgebra in the (fund.) \otimes (fund.) representation reads

$$R_h|_{(fund. \otimes fund.)} = \begin{pmatrix} q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & q - q^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & q - q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & q - q^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -q^{-2} \end{pmatrix}. \quad (2.8)$$

The \mathcal{R}_h -matrix in the $(fund. \otimes fund.)$ representation is obtained, from (2.9), in the following manner:

$$\begin{aligned} R_h|_{(fund. \otimes fund.)} &= \lim_{q \rightarrow 1} \left[E_q^{-1} \left(\frac{h\hat{e}_1}{q-1} \right)_{fund.} \otimes E_q^{-1} \left(\frac{h\hat{e}_1}{q-1} \right)_{fund.} \right] R_q \left[E_q \left(\frac{h\hat{e}_1}{q-1} \right)_{fund.} \otimes E_q \left(\frac{h\hat{e}_1}{q-1} \right)_{fund.} \right] \\ &= \begin{pmatrix} 1 & h & 0 & -h & h^2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & h & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -h & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \end{aligned} \quad (2.9)$$

Similarly, using a Maple program[†], we obtain, for (fundamental irrep.) \otimes (arbitrary irrep.), the following expression:

$$L \equiv R_{\mathbf{h}}|_{(fund.\otimes arb.)} = \begin{pmatrix} T & -\mathbf{h}H_1 + \frac{\mathbf{h}}{2}(T - T^{-1}) & 0 \\ 0 & T^{-1} & 0 \\ 0 & 0 & (-1)^F \end{pmatrix}, \quad (2.10)$$

where

$$H_1 = \frac{1}{2}(T + T^{-1})h_1 = \sqrt{1 + \mathbf{h}^2 e_1^2}h_1, \quad T^{\pm 1} = \pm \mathbf{h}e_1 + \sqrt{1 + \mathbf{h}^2 e_1^2}. \quad (2.11)$$

The above L operator allows immediate construction of the full Hopf structure of the Borel subalgebra of the $\mathcal{U}_{\mathbf{h}}(sl(2|1))$ algebra via the FRT formalism[‡].

2.2 Nonlinear map and Hopf structure

Following refs. [4, 6], let us introduce the generator

$$F_1 = f_1 - \frac{\mathbf{h}^2}{4}e_1(h_1^2 - 1). \quad (2.12)$$

We then show that

$$\begin{aligned} TT^{-1} &= T^{-1}T = 1, & [H_1, T^{\pm 1}] &= T^{\pm 2} - 1, \\ [T^{\pm 1}, F_1] &= \pm \frac{\mathbf{h}}{2}(H_1 T^{\pm 1} + T^{\pm 1} H_1), & [H_1, F_1] &= -\frac{1}{2}(TF_1 + F_1 T + T^{-1}F_1 + F_1 T^{-1}), \end{aligned} \quad (2.13)$$

with the well known coproducts, counits and antipodes [22]

$$\begin{aligned} \Delta(H_1) &= H_1 \otimes T + T^{-1} \otimes H_1, & \Delta(T^{\pm 1}) &= T^{\pm 1} \otimes T^{\pm 1}, & \Delta(F_1) &= F_1 \otimes T + T^{-1} \otimes F_1, \\ S(H_1) &= -TH_1T^{-1}, & S(T^{\pm 1}) &= T^{\mp 1}, & S(F_1) &= -TF_1T^{-1}, \\ \epsilon(H_1) &= \epsilon(F_1) = 0, & \epsilon(T^{\pm 1}) &= 1. \end{aligned} \quad (2.14)$$

This implies that the Ohn's structure follows from the bosonic generators $\{h_1, e_1, f_1\}$. The algebraic properties (2.11) and (2.12) exhibits clearly the embedding of $\mathcal{U}_{\mathbf{h}}(sl(2))$ in $\mathcal{U}_{\mathbf{h}}(sl(2|1))$.

To complete now the $\mathcal{U}_{\mathbf{h}}(sl(2|1))$ superalgebra, we introduce the following \mathbf{h} -deformed fermionic root generators:

$$\begin{aligned} H_2 &= h_2 - \frac{\mathbf{h}^2}{2}e_1^2h_1, & E_2 &= e_2 - \frac{\mathbf{h}^2}{4}e_1e_3(2h_1 + 1), & F_2 &= f_2, \\ H_3 &= h_3 + \frac{\mathbf{h}^2}{2}e_1^2h_1, & E_3 &= e_3, & F_3 &= f_3 + \frac{\mathbf{h}^2}{4}e_1f_2(2h_1 + 1). \end{aligned} \quad (2.15)$$

The generators E_2, E_3, F_2 and F_3 are odd, while H_2 and H_3 are even. The expressions (2.11), (2.12) and (2.15) define a realization of the super-Jordanian subalgebra $\mathcal{U}_{\mathbf{h}}(sl(2|1))$ with the classical generators via a nonlinear map (*Other invertible maps relating the super-Jordanian and the classical generators may also be considered*) and permit immediate explicit construction of the finite-dimensional irreducible representations of the $\mathcal{U}_{\mathbf{h}}(sl(2|1))$ superalgebra. In the followings we only quote the final results:

Proposition 1 *The nonstandard (super-Jordanian) enveloping superalgebra $\mathcal{U}_{\mathbf{h}}(sl(2|1))$ is an associative superalgebra over \mathbb{C} generated by $\{H_1, T, T^{-1}, F_1, H_2, E_2, F_2, H_3, E_3, F_3\}$ satisfying, along with (2.15) and*

[†]Our program was performed for (fund.) \otimes (fund.), (fund.) \otimes (vect.), etc.

[‡]The algebraic and coalgebraic properties of the Borel subalgebra are respectively given by $R_{\mathbf{h}}|_{(fund.\otimes fund.)} L_1 L_2 = L_2 L_1$, $R_{\mathbf{h}}|_{(fund.\otimes fund.)} \Delta(L) = L \otimes L$, $\varepsilon(L) = 1$ and $S(L) = L^{-1}$.

(2.17), the commutation relations

$$\begin{aligned}
[H_1, H_2] &= -\frac{1}{4} (T - T^{-1})^2 H_1, & [H_1, H_3] &= \frac{1}{4} (T - T^{-1})^2 H_1, & [H_2, H_3] &= 0, \\
[H_1, E_2] &= -\frac{1}{2} (T + T^{-1}) E_2 - \frac{\hbar}{2} (T - T^{-1}) E_3 H_1 - \frac{\hbar}{4} (T^2 - T^{-2}) E_3, \\
[H_1, F_3] &= -\frac{1}{2} (T + T^{-1}) F_3 + \frac{\hbar}{2} (T - T^{-1}) F_2 H_1 + \frac{\hbar}{4} (T^2 - T^{-2}) F_2, \\
[H_1, F_2] &= \frac{1}{2} (T + T^{-1}) F_2, & [H_1, E_3] &= \frac{1}{2} (T + T^{-1}) E_3, \\
[H_2, T^{\pm 1}] &= -\frac{1}{4} (T^{\pm 3} - T^{\mp 1}), & [H_3, T^{\pm 1}] &= \frac{1}{4} (T^{\pm 3} - T^{\mp 1}), \\
[H_2, F_1] &= \frac{1}{4} (T + T^{-1})^2 F_1 - \frac{\hbar}{4} (T - T^{-1}) H_1^2 - \frac{\hbar}{4} (T^2 - T^{-2}) H_1 - \frac{\hbar}{16} (T^2 - T^{-2}) (T + T^{-1}), \\
[H_3, F_1] &= -\frac{1}{4} (T + T^{-1})^2 F_1 + \frac{\hbar}{4} (T - T^{-1}) H_1^2 + \frac{\hbar}{4} (T^2 - T^{-2}) H_1 + \frac{\hbar}{16} (T^2 - T^{-2}) (T + T^{-1}), \\
[H_2, E_2] &= \frac{\hbar}{16} (T + T^{-1}) (T^2 - T^{-2}) E_3 + \frac{1}{8} (T - T^{-1})^2 E_2, \\
[H_3, F_3] &= \frac{\hbar}{16} (T - T^{-1}) (T^2 - T^{-2}) F_2 - \frac{1}{8} (T - T^{-1})^2 F_3, \\
[H_2, F_3] &= \frac{1}{8} (T^2 + 6 + T^{-2}) F_3 - \frac{\hbar}{16} (T^2 - T^{-2}) (T + T^{-1}) F_2, \\
[H_3, E_2] &= -\frac{1}{8} (T^2 + 6 + T^{-2}) E_2 - \frac{\hbar}{16} (T^2 - T^{-2}) (T + T^{-1}) E_3, \\
[H_2, F_2] &= -\frac{1}{8} (T - T^{-1})^2 F_2, & [H_3, E_3] &= \frac{1}{8} (T - T^{-1})^2 E_3, \\
[H_3, F_2] &= \frac{1}{8} (T^2 + 6 + T^{-2}) F_2, & [H_2, E_3] &= -\frac{1}{8} (T^2 + 6 + T^{-2}) E_3, \\
[E_2, F_2] &= H_2 - \frac{1}{16} (T - T^{-1})^2 - \frac{\hbar}{4} (T - T^{-1}) E_3 F_2, \\
[E_3, F_3] &= H_3 + \frac{1}{16} (T - T^{-1})^2 + \frac{\hbar}{4} (T - T^{-1}) F_2 E_3, \\
[T^{\pm 1}, F_2] &= [T^{\pm 1}, E_3] = 0, & F_2^2 &= E_3^2 = 0, & [F_2, F_1] &= F_3, & [F_1, E_3] &= E_2, \\
E_2^2 &= \frac{\hbar}{4} (T - T^{-1}) E_3 E_2, & F_3^2 &= -\frac{\hbar}{4} (T - T^{-1}) F_2 F_3, & [E_2, E_3] &= [F_2, F_3] = 0, \\
[T^{\pm 1}, E_2] &= \pm \frac{\hbar}{2} (T^{\pm 2} + 1) E_3, & [T^{\pm 1}, F_3] &= \mp \frac{\hbar}{2} (T^{\pm 2} + 1) F_2, & [F_2, E_3] &= \frac{1}{2\hbar} (T - T^{-1}), \\
[E_2, F_1] &= \frac{\hbar}{4} (T - T^{-1}) E_2 + \frac{\hbar}{2} (T - T^{-1}) E_3 F_1 - \frac{\hbar^2}{4} E_3 H_1^2 - \frac{3\hbar^2}{8} (T + T^{-1}) E_3 H_1 - \frac{\hbar^2}{2} E_3 \\
&\quad - \frac{15\hbar^2}{64} (T - T^{-1})^2 E_3, \\
[F_3, F_1] &= \frac{\hbar}{4} (T - T^{-1}) F_3 - \frac{\hbar}{2} (T - T^{-1}) F_2 F_1 + \frac{\hbar^2}{4} F_2 H_1^2 + \frac{3\hbar^2}{8} (T + T^{-1}) F_2 H_1 + \frac{\hbar^2}{2} F_2 \\
&\quad + \frac{15\hbar^2}{64} (T - T^{-1})^2 F_2, \\
[F_3, E_2] &= F_1 - \frac{\hbar}{4} (T - T^{-1}) F_2 E_2 + \frac{\hbar}{4} (T - T^{-1}) E_3 F_3 - \frac{\hbar}{8} (T - T^{-1}) H_1^2 - \frac{\hbar}{8} (T^2 - T^{-2}) H_1 \\
&\quad - \frac{\hbar}{16} H_1 (T^2 - T^{-2}) - \frac{7\hbar}{128} (T - T^{-1})^3.
\end{aligned} \tag{2.16}$$

The \mathbb{Z}_2 -grading in $\mathcal{U}_{\hbar}(sl(2|1))$ is uniquely defined by the requirement that the only odd generators are E_2, F_2, E_3 and F_3 . It is obvious that as $\hbar \rightarrow 0$, we have $(E_2, F_2, H_2, E_3, F_3, H_3) \rightarrow (e_2, f_2, h_2, e_3, f_3, h_3)$.

Proposition 2 Let us note that there exist a \mathbb{C} -algebra automorphism of $\mathcal{U}_{\hbar}(sl(2|1))$ such that

$$\Phi \left(T^{\pm 1}, F_1, H_1, E_2, F_2, H_2, E_3, F_3, H_3 \right) \longrightarrow \left(T^{\pm 1}, F_1, H_1, F_3, -E_3, -H_3, -F_2, E_2, -H_2 \right). \tag{2.17}$$

(For $\hbar = 0$, this automorphism reduces to (2.5)).

Proposition 3 *The nonstandard (super-Jordanian) quantum enveloping superalgebra $\mathcal{U}_\hbar(\mathfrak{sl}(2|1))$ admits a Hopf structure with coproducts, antipodes and counits determined by (2.15) and*

$$\begin{aligned}
\Delta(E_2) &= E_2 \otimes T^{1/2} + T^{-1/2} \otimes E_2 + \frac{\hbar}{4} T^{-1} E_3 \otimes (T^{-1/2} H_1 + H_1 T^{-1/2}) - \frac{\hbar}{4} (T^{1/2} H_1 + H_1 T^{1/2}) \otimes T E_3, \\
\Delta(F_2) &= F_2 \otimes T^{-1/2} + T^{1/2} \otimes F_2, \quad \Delta(E_3) = E_3 \otimes T^{-1/2} + T^{1/2} \otimes E_3, \\
\Delta(F_3) &= F_3 \otimes T^{1/2} + T^{-1/2} \otimes F_3 - \frac{\hbar}{4} T^{-1} F_2 \otimes (T^{-1/2} H_1 + H_1 T^{-1/2}) + \frac{\hbar}{4} (T^{1/2} H_1 + H_1 T^{1/2}) \otimes T F_2, \\
\Delta(H_2) &= H_2 \otimes 1 + 1 \otimes H_2 + \frac{1}{4} T H_1 \otimes (1 - T^2) + \frac{1}{4} (1 - T^{-2}) \otimes T^{-1} H_1, \\
\Delta(H_3) &= H_3 \otimes 1 + 1 \otimes H_3 - \frac{1}{4} T H_1 \otimes (1 - T^2) - \frac{1}{4} (1 - T^{-2}) \otimes T^{-1} H_1, \\
S(E_2) &= -E_2 - \frac{\hbar}{2} (T + T^{-1}) E_3, \quad S(F_3) = -F_3 + \frac{\hbar}{2} (T + T^{-1}) F_2, \\
S(F_2) &= -F_2, \quad S(E_3) = -E_3, \\
S(H_2) &= -H_2 + \frac{1}{2} (T^{-2} - 1), \quad S(H_3) = -H_3 - \frac{1}{2} (T^{-2} - 1), \\
\epsilon(H_2) &= \epsilon(H_3) = \epsilon(E_2) = \epsilon(F_2) = \epsilon(E_3) = \epsilon(F_3) = 0.
\end{aligned} \tag{2.18}$$

All the Hopf superalgebra axioms can be verified by direct calculations. We remark that our coproducts have simpler forms compared to those given in the literature. This is one main advantage of our procedure.

Proposition 4 *The universal \mathcal{R}_\hbar -matrix of $\mathcal{U}_\hbar(\mathfrak{sl}(2|1))$ has the following form:*

$$\mathcal{R}_\hbar = \exp\left(-\hbar X_1 \otimes T H_1\right) \exp\left(\hbar T H_1 \otimes X_1\right), \tag{2.19}$$

where $X_1 = \hbar^{-1} \ln T$. The element (2.19) coincides with the pure $\mathcal{U}_\hbar(\mathfrak{sl}(2))$ universal \mathcal{R}_\hbar -matrix [19].

3 $\mathcal{U}(\mathfrak{sl}(3|1))$: Nonstandard quantization and Hopf Structure

The major interest of our approach is that it can be generalized for obtaining super-Jordanian quantum superalgebras $\mathcal{U}_\hbar(\mathfrak{sl}(N|1))$ of higher dimensions. We start here with $\mathcal{U}_\hbar(\mathfrak{sl}(3|1))$. In our notations e_{ij} is an $(N+1) \times (N+1)$ matrix with only the (i, j) matrix element being equal to 1, all other matrix elements are zero. Let $h_{12} = e_{11} - e_{22}$, $h_{23} = e_{22} - e_{33}$, $h_{34} = e_{33} + e_{44}$, e_{12} , e_{23} , e_{34} , e_{21} , e_{32} and e_{43} be the standard Chevalley generators of $\mathcal{U}(\mathfrak{sl}(3|1))$. The generators h_{12} , h_{23} , e_{12} , e_{23} , e_{21} , e_{32} , and h_{34} are even, while e_{34} and e_{43} are odd. The generators corresponding to the other roots, obtained by action of the Weyl group, are denoted by $e_{13} = [e_{12}, e_{23}]$, $e_{14} = [e_{13}, e_{34}]$, $e_{24} = [e_{23}, e_{34}]$, $e_{31} = [e_{32}, e_{21}]$, $e_{41} = [e_{43}, e_{31}]$, $e_{42} = [e_{43}, e_{32}]$, $h_{13} = e_{11} - e_{33} \equiv h_{12} + h_{23}$, $h_{14} = e_{11} + e_{44} \equiv h_{13} + h_{34}$ and $h_{24} = e_{22} + e_{44} \equiv h_{23} + h_{34}$ [§]. The commutator $[\cdot, \cdot]$ is understood as the \mathbb{Z}_2 -graded one, i.e.

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - (-)^{\deg(e_{ij}) \deg(e_{kl})} \delta_{li} e_{kj}. \tag{3.1}$$

There exist a \mathbb{C} -algebra automorphism ϕ of $\mathcal{U}(\mathfrak{sl}(3|1))$ such that

$$\phi(e_{12}, e_{21}, h_{12}, e_{23}, e_{32}, h_{23}, e_{34}, e_{43}, h_{34}, \dots) \longrightarrow (e_{23}, e_{32}, h_{23}, e_{12}, e_{12}, h_{12}, e_{41}, e_{14}, -h_{14}, \dots) \tag{3.2}$$

[§]The elements $\{h_{12}, h_{23}, e_{12}, e_{23}, e_{21}, e_{32}, e_{13}, e_{31}, h_{13}\}$ build here the subalgebra $\mathcal{U}(\mathfrak{sl}(3))$ of $\mathcal{U}(\mathfrak{sl}(3|1))$.

3.1 The Bosonic part: $\mathcal{U}_h(sl(3))$ subalgebra

As in the $\mathcal{U}_h(sl(2|1))$ superalgebra, the super-Jordanian deformation arises here from the bosonic generators corresponding to the higher root, i.e. e_{13} , e_{31} and h_{13} . These generators are deformed as follows[¶]:

$$T^{\pm 1} = \pm h e_{13} + \sqrt{1 + h^2 e_{13}^2}, \quad H_{13} = \sqrt{1 + h^2 e_{13}^2} h_{13}, \quad E_{31} = e_{31} - \frac{h^2}{4} e_{13} (h_{13}^2 - 1). \quad (3.3)$$

To complete first the $\mathcal{U}_h(sl(3)) \subset \mathcal{U}_h(sl(3|1))$ subalgebra (*the bosonic part of $\mathcal{U}_h(sl(3|1))$*), let us introduce the following h -deformed generators:

$$\begin{aligned} H_{12} &= h_{12} + \frac{h^2}{2} e_{13}^2 h_{13}, & E_{12} &= e_{12}, & E_{21} &= e_{21} + \frac{h^2}{4} e_{23} e_{13} (2h_{13} + 1), \\ H_{23} &= h_{23} + \frac{h^2}{2} e_{13}^2 h_{13}, & E_{23} &= e_{23}, & E_{32} &= e_{32} - \frac{h^2}{4} e_{12} e_{13} (2h_{13} + 1), \end{aligned} \quad (3.4)$$

where it is obvious that as $h \rightarrow 0$, we have $(H_{12}, E_{12}, E_{21}, H_{23}, E_{23}, E_{32}, ; H_{13}, T, T^{-1}, E_{31}) \rightarrow (h_{12}, e_{12}, e_{21}, h_{23}, e_{23}, e_{32}, h_{13}, 1, 1, e_{31})$. The expressions (3.3) and (3.4) define a realization of the Jordanian subalgebra $\mathcal{U}_h(sl(3))$ embedded in $\mathcal{U}_h(sl(3|1))$ with the classical generators via a nonlinear map. Another map has been considered in [6]. Our construction leads to the following results:

Proposition 5 *The generating elements $\{H_{12}, E_{12}, E_{21}, H_{23}, E_{23}, E_{32}, H_{13}, T, T^{-1}, E_{31}\}$ of the Jordanian quantum algebra $\mathcal{U}_h(sl(3))$ obey the following commutations rules:*

$$\begin{aligned} TT^{-1} &= T^{-1}T = 1, & [H_{13}, T^{\pm 1}] &= T^{\pm 2} - 1, & [T^{\pm 1}, E_{31}] &= \pm \frac{h}{2} (H_{13} T^{\pm 1} + T^{\pm 1} H_{13}), \\ [H_{13}, E_{31}] &= -\frac{1}{2} \left((T + T^{-1}) E_{31} + E_{31} (T + T^{-1}) \right), \\ [H_{12}, H_{23}] &= 0, & [H_{12}, H_{13}] &= -\frac{1}{4} (T - T^{-1})^2 H_{13}, & [H_{23}, H_{13}] &= -\frac{1}{4} (T - T^{-1})^2 H_{13}, \\ [H_{12}, E_{12}] &= 2E_{12} + \frac{1}{8} (T - T^{-1})^2 E_{12}, & [H_{12}, E_{23}] &= -E_{23} + \frac{1}{8} (T - T^{-1})^2 E_{23}, \\ [H_{23}, E_{12}] &= -E_{12} + \frac{1}{8} (T - T^{-1})^2 E_{12}, & [H_{23}, E_{23}] &= 2E_{23} + \frac{1}{8} (T - T^{-1})^2 E_{23}, \\ [H_{12}, E_{21}] &= -2E_{21} - \frac{1}{8} (T - T^{-1})^2 E_{21} + \frac{h}{16} (T + T^{-1}) (T^2 - T^{-2}) E_{23}, \\ [H_{23}, E_{32}] &= -2E_{32} - \frac{1}{8} (T - T^{-1})^2 E_{32} - \frac{h}{16} (T + T^{-1}) (T^2 - T^{-2}) E_{12}, \\ [H_{12}, E_{32}] &= E_{32} - \frac{1}{8} (T - T^{-1})^2 E_{32} - \frac{h}{16} (T + T^{-1}) (T^2 - T^{-2}) E_{12}, \\ [H_{23}, E_{21}] &= E_{21} - \frac{1}{8} (T - T^{-1})^2 E_{21} + \frac{h}{16} (T + T^{-1}) (T^2 - T^{-2}) E_{23}, \\ [H_{13}, E_{12}] &= \frac{1}{2} (T + T^{-1}) E_{12}, & [H_{13}, E_{23}] &= \frac{1}{2} (T + T^{-1}) E_{23}, \\ [H_{13}, E_{21}] &= -\frac{1}{2} (T + T^{-1}) E_{21} + \frac{h}{2} (T - T^{-1}) E_{23} H_{13} + \frac{h}{4} (T^2 - T^{-2}) E_{23}, \\ [H_{13}, E_{32}] &= -\frac{1}{2} (T + T^{-1}) E_{32} - \frac{h}{2} (T - T^{-1}) E_{12} H_{13} - \frac{h}{4} (T^2 - T^{-2}) E_{12}, \end{aligned}$$

[¶]Similarly to Ref. [6], by applying the contraction process on the R_q -matrix in the $(fund. \otimes arb.)$, associated to $\mathcal{U}_q(sl(3|1))$, we obtain:

$$R_h|_{(fund. \otimes arb.)} = \begin{pmatrix} T & 2hT^{-1/2}e_{23} & -\frac{h}{2}(T+T^{-1})(h_1+h_2) + \frac{h}{2}(T-T^{-1}) & 0 \\ 0 & I & -2hT^{1/2}e_{12} & 0 \\ 0 & 0 & T & 0 \\ 0 & 0 & 0 & (-1)^F \end{pmatrix}.$$

$$\begin{aligned}
[E_{21}, F_{31}] &= \frac{\hbar}{4} (T - T^{-1}) E_{21} - \frac{\hbar}{2} (T - T^{-1}) E_{23} E_{31} + \frac{\hbar^2}{4} E_{23} H_{13}^2 + \frac{3\hbar^2}{8} (T + T^{-1}) E_{23} H_{13} \\
&\quad + \frac{\hbar^2}{2} E_{23} + \frac{15\hbar^2}{64} (T - T^{-1})^2 E_{23}, \\
[E_{32}, F_{31}] &= \frac{\hbar}{4} (T - T^{-1}) E_{32} + \frac{\hbar}{2} (T - T^{-1}) E_{12} E_{31} - \frac{\hbar^2}{4} E_{12} H_{13}^2 - \frac{3\hbar^2}{8} (T + T^{-1}) E_{12} H_{13} \\
&\quad - \frac{\hbar^2}{2} E_{12} - \frac{15\hbar^2}{64} (T - T^{-1})^2 E_{12}, \\
[H_{12}, T^{\pm 1}] &= -\frac{1}{4} (T^{\pm 3} - T^{\mp 1}), \quad [H_{23}, T^{\pm 1}] = -\frac{1}{4} (T^{\pm 3} - T^{\mp 1}), \\
[H_{12}, E_{31}] &= -\frac{1}{4} (T + T^{-1})^2 E_{31} + \frac{\hbar}{4} (T - T^{-1}) H_{13}^2 + \frac{\hbar}{2} (T + T^{-1}) E_{23} H_{13} + \frac{\hbar}{16} (T^3 + T - T^{-1} - T^{-3}), \\
[H_{23}, E_{31}] &= -\frac{1}{4} (T + T^{-1})^2 E_{31} + \frac{\hbar}{4} (T - T^{-1}) H_{13}^2 - \frac{\hbar}{2} (T + T^{-1}) E_{12} H_{13} + \frac{\hbar}{16} (T^3 + T - T^{-1} - T^{-3}), \\
[F_{32}, E_{21}] &= F_{31} + \frac{\hbar}{4} (T - T^{-1}) (E_{12} E_{21} + E_{23} E_{32}) - \frac{\hbar}{8} (T - T^{-1}) H_{13}^2 - \frac{\hbar}{4} (T - T^{-1}) \\
&\quad - \frac{3\hbar}{16} (T^2 - T^{-2}) H_{13} - \frac{9\hbar}{128} (T - T^{-1})^3, \\
[E_{12}, E_{21}] &= H_{12} + \frac{1}{16} (T - T^{-1})^2 - \frac{\hbar}{4} (T - T^{-1}) E_{23} E_{12}, \\
[E_{23}, E_{32}] &= H_{23} + \frac{1}{16} (T - T^{-1})^2 + \frac{\hbar}{4} (T - T^{-1}) E_{12} E_{23}, \quad [T^{\pm 1}, E_{12}] = [T^{\pm 1}, E_{23}] = 0, \\
[E_{23}, E_{21}] &= -\frac{\hbar}{4} (T - T^{-1}) E_{23}^2, \quad [E_{12}, E_{32}] = \frac{\hbar}{4} (T - T^{-1}) E_{12}^2, \\
[T^{\pm 1}, E_{21}] &= \mp \frac{\hbar}{2} (T^{\pm 2} + 1) E_{23}, \quad [T^{\pm 1}, E_{32}] = \pm \frac{\hbar}{2} (T^{\pm 2} + 1) E_{12}, \quad [E_{12}, E_{23}] = \frac{1}{2\hbar} (T - T^{-1}).
\end{aligned} \tag{3.5}$$

The other commutators remain undeformed.

3.2 The Fermionic part

To describe the fermionic part of the super-Jordanian quantum superalgebra $\mathcal{U}_{\hbar}(sl(3|1))$, we define the elements

$$\begin{aligned}
H_{34} &= h_{34} - \frac{\hbar^2}{2} e_{13}^2 h_{13}, \quad E_{34} = e_{34} - \frac{\hbar^2}{4} e_{13} e_{14} (2h_{13} + 1), \quad E_{43} = e_{43}, \\
H_{24} &= h_{24}, \quad E_{24} = e_{24}, \quad E_{42} = e_{42}, \\
H_{14} &= h_{14} + \frac{\hbar^2}{2} e_{13}^2 h_{13}, \quad E_{14} = e_{14}, \quad E_{41} = e_{41} + \frac{\hbar^2}{4} e_{13} e_{43} (2h_{13} + 1).
\end{aligned} \tag{3.6}$$

The generators E_{34} , E_{43} , E_{24} , E_{42} , E_{14} and E_{41} are odd, while H_{34} , H_{24} and H_{14} are even. The expressions (3.3), (3.4) and (3.6) constitute a nonlinear realization of the super-Jordanian quantum superalgebra $\mathcal{U}_{\hbar}(sl(3|1))$ with the classical generators. Let us just remark that the \mathbb{C} -algebra automorphism (3.2) can be easily extended to our construction, i.e.

$$\phi(E_{12}, E_{21}, H_{12}, E_{23}, E_{32}, H_{23}, E_{34}, E_{43}, H_{34}, \dots) \longrightarrow (E_{23}, E_{32}, H_{23}, E_{12}, E_{12}, H_{12}, E_{41}, E_{14}, -H_{14}, \dots) \tag{3.7}$$

Proposition 6 *The super-Jordanian quantum superalgebra $\mathcal{U}_{\hbar}(sl(3|1))$ is then an associative superalgebra over \mathbb{C} spanned by $\{H_{12}, E_{12}, E_{21}, H_{23}, E_{23}, E_{32}, H_{13}, T, T^{-1}, E_{31}, H_{34}, E_{34}, E_{43}, H_{24}, E_{24}, E_{42}, H_{14}, E_{14}, E_{41}\}$, satisfying along with (3.5), the commutation relations (we list here only the deformed commutator)*

$$\begin{aligned}
[H_{13}, H_{34}] &= -\frac{1}{4} (T - T^{-1})^2 H_{13}, \quad [H_{13}, H_{14}] = \frac{1}{4} (T - T^{-1})^2 H_{13}, \\
[H_{13}, E_{14}] &= \frac{1}{2} (T + T^{-1}) E_{14}, \quad [H_{13}, E_{43}] = \frac{1}{2} (T + T^{-1}) E_{43}, \\
[H_{13}, E_{41}] &= -\frac{1}{2} (T + T^{-1}) E_{41} + \frac{\hbar}{2} (T - T^{-1}) E_{43} H_{13} + \frac{\hbar}{4} (T^2 - T^{-2}) E_{43},
\end{aligned}$$

$$\begin{aligned}
[H_{13}, E_{34}] &= -\frac{1}{2} (T + T^{-1}) E_{34} - \frac{\hbar}{2} (T - T^{-1}) E_{14} H_{13} - \frac{\hbar}{2} (T^2 - T^{-2}) E_{14}, \\
[H_{34}, E_{14}] &= -\left(1 + \frac{1}{8} (T - T^{-1})^2\right) E_{14}, \quad [H_{14}, E_{43}] = \left(1 + \frac{1}{8} (T - T^{-1})^2\right) E_{43}, \\
[H_{34}, E_{41}] &= \left(1 + \frac{1}{8} (T - T^{-1})^2\right) E_{41} - \frac{\hbar}{16} (T^2 - T^{-2}) (T + T^{-1}) E_{43}, \\
[H_{34}, E_{34}] &= \frac{1}{8} (T - T^{-1})^2 E_{34} + \frac{\hbar}{16} (T^2 - T^{-2}) (T - T^{-1}) E_{14}, \\
[H_{34}, E_{43}] &= -\frac{1}{8} (T - T^{-1})^2 E_{43}, \\
[H_{34}, T^{\pm 1}] &= -\frac{1}{4} (T^{\pm 3} - T^{\mp 1}), \quad [H_{14}, T^{\pm 1}] = \frac{1}{4} (T^{\pm 3} - T^{\mp 1}), \\
[H_{34}, E_{31}] &= \frac{1}{4} (T + T^{-1})^2 E_{31} - \frac{\hbar}{4} (T - T^{-1}) H_{13}^2 - \frac{\hbar}{4} (T^2 - T^{-2}) H_{13} - \frac{\hbar}{16} (T^2 - T^{-2}) (T + T^{-1}), \\
[H_{14}, E_{31}] &= -\frac{1}{4} (T + T^{-1})^2 E_{31} + \frac{\hbar}{4} (T - T^{-1}) H_{13}^2 + \frac{\hbar}{4} (T^2 - T^{-2}) H_{13} + \frac{\hbar}{16} (T^2 - T^{-2}) (T + T^{-1}), \\
[H_{14}, E_{34}] &= -\left(1 + \frac{1}{8} (T - T^{-1})^2\right) E_{34} - \frac{\hbar}{16} (T^2 - T^{-2}) (T + T^{-1}) E_{43}, \\
[T^{\pm 1}, E_{34}] &= \pm \frac{\hbar}{2} (T^{\pm 2} + 1) E_{14}, \quad [T^{\pm 1}, E_{41}] = \mp \frac{\hbar}{2} (T^{\pm 2} + 1) E_{43}, \\
[E_{43}, E_{14}] &= \frac{1}{2\hbar} (T - T^{-1}), \\
[E_{34}, E_{43}] &= H_{34} - \frac{1}{16} (T - T^{-1})^2 - \frac{\hbar}{4} (T - T^{-1}) E_{14} E_{43}, \\
[E_{14}, E_{41}] &= H_{14} + \frac{1}{16} (T - T^{-1})^2 + \frac{\hbar}{4} (T - T^{-1}) E_{43} E_{14}, \\
[E_{43}, E_{31}] &= \frac{\hbar}{4} (T - T^{-1}) E_{34} + \frac{\hbar}{2} (T - T^{-1}) E_{14} E_{31} - \frac{\hbar^2}{4} E_{14} H_{13}^2 - \frac{3\hbar^2}{8} (T + T^{-1}) E_{14} H_{13} \\
&\quad - \frac{\hbar^2}{2} E_{14} - \frac{15\hbar^2}{64} (T - T^{-1})^2 E_{14}, \\
[E_{41}, E_{31}] &= \frac{\hbar}{4} (T - T^{-1}) E_{41} - \frac{\hbar}{2} (T - T^{-1}) E_{43} E_{31} + \frac{\hbar^2}{4} E_{43} H_{13}^2 + \frac{3\hbar^2}{8} (T + T^{-1}) E_{43} H_{13} \\
&\quad + \frac{\hbar^2}{2} E_{43} + \frac{15\hbar^2}{64} (T - T^{-1})^2 E_{43}, \\
[E_{43}, E_{32}] &= F_{42} + \frac{\hbar}{4} (T - T^{-1}) E_{12} E_{43}, \\
E_{34}^2 &= \frac{\hbar}{4} (T - T^{-1}) E_{14} E_{34}, \quad E_{41}^2 = -\frac{\hbar}{4} (T - T^{-1}) E_{43} E_{41}, \\
[T^{\pm 1}, E_{14}] &= 0, \quad [T^{\pm 1}, E_{43}] = 0, \quad [T^{\pm 1}, E_{24}] = 0, \quad [T^{\pm 1}, E_{42}] = 0, \\
[E_{34}, E_{41}] &= F_{31} - \frac{\hbar}{4} (T - T^{-1}) E_{43} E_{34} + \frac{\hbar}{4} (T - T^{-1}) E_{14} F_{41} - \frac{\hbar}{8} (T - T^{-1}) H_{13}^2 - \frac{\hbar}{8} (T^2 - T^{-2}) H_{13}^2 \\
&\quad - \frac{\hbar}{16} H_{13} (T^2 - T^{-2}) + \frac{7\hbar}{128} (T - T^{-1})^3. \tag{3.8}
\end{aligned}$$

The coalgebraic structure will be presented, for the general case, in the following section.

4 $\mathcal{U}(sl(N|1))$: Generalization

From the above studies, it is easy to see that:

Proposition 7 *The superalgebra $\mathcal{U}_{\hbar}(sl(N|1))$ can be realized via the nonlinear map:*

$$\begin{aligned}
T^{\pm 1} &= \pm \hbar e_{1N} + \sqrt{1 + \hbar^2 e_{1N}^2}, \quad H_{1N} = \sqrt{1 + \hbar^2 e_{1N}^2} h_{1N}, \quad E_{N1} = e_{N1} - \frac{\hbar^2}{4} e_{1N} (h_{1N}^2 - 1), \\
H_{ij} &= h_{ij} + \frac{\hbar^2}{2} (\delta_{i1} + \delta_{jN}) e_{1N}^2 h_{1N}, \quad i < j \in \{1, 2, \dots, N\} \text{ and } (i, j) \neq (1, N),
\end{aligned}$$

$$\begin{aligned}
E_{ij} &= e_{ij}, & i < j \in \{1, 2, \dots, N\} \text{ and } (i, j) \neq (1, N), \\
E_{ji} &= e_{ji} + \frac{\hbar^2}{4} (\delta_{i1} e_{jN} - \delta_{Nj} e_{1i}) (2h_{1N} + 1), & i < j \in \{1, 2, \dots, N\} \text{ and } (i, j) \neq (1, N), \\
H_{i,N+1} &= h_{i,N+1} + \frac{\hbar^2}{2} (\delta_{i1} - \delta_{iN}) e_{1N}^2 h_{1N}, & i \in \{1, 2, \dots, N\}, \\
E_{i,N+1} &= e_{i,N+1} - \frac{\hbar^2}{4} \delta_{iN} e_{1,N+1} e_{1N} (2h_{1N} + 1), & i \in \{1, 2, \dots, N\}, \\
E_{N+1,i} &= e_{N+1,i} + \frac{\hbar^2}{4} \delta_{iN} e_{N+1,N} e_{1N} (2h_{1N} + 1), & i \in \{1, 2, \dots, N\},
\end{aligned} \tag{4.1}$$

with the coproducts

$$\begin{aligned}
\Delta(H_{1N}) &= H_{1N} \otimes T + T^{-1} \otimes H_{1N}, & \Delta(T^{\pm 1}) &= T^{\pm 1} \otimes T^{\pm 1}, & \Delta(E_{N1}) &= E_{N1} \otimes T + T^{-1} \otimes E_{N1}, \\
\Delta(H_{ij}) &= H_{ij} \otimes 1 + 1 \otimes H_{ij} - \frac{1}{4} (\delta_{i1} + \delta_{jN}) \left(TH_{1N} \otimes (1 - T^2) + (1 - T^{-2}) \otimes T^{-1} H_{1N} \right), \\
& & i < j \in \{1, 2, \dots, N\} \text{ and } (i, j) \neq (1, N), \\
\Delta(E_{ij}) &= E_{ij} \otimes T^{-(\delta_{i1} + \delta_{jN})/2} + T^{(\delta_{i1} + \delta_{jN})/2} \otimes E_{ij}, & i < j \in \{1, 2, \dots, N\} \text{ and } (i, j) \neq (1, N), \\
\Delta(E_{ji}) &= E_{ji} \otimes T^{(\delta_{i1} + \delta_{jN})/2} + T^{-(\delta_{i1} + \delta_{jN})/2} \otimes E_{ji} + \frac{\hbar}{4} T^{-1} (-\delta_{i1} E_{jN} + \delta_{jN} E_{1i}) \otimes \left(T^{-1/2} H_{1N} + \right. \\
& & \left. H_{1N} T^{-1/2} \right) - \frac{\hbar}{4} \left(T^{1/2} H_{1N} + H_{1N} T^{1/2} \right) \otimes T (-\delta_{i1} E_{jN} + \delta_{jN} E_{1i}) \\
& & i < j \in \{1, 2, \dots, N\} \text{ and } (i, j) \neq (1, N), \\
\Delta(H_{i,N+1}) &= H_{i,N+1} \otimes 1 + 1 \otimes H_{i,N+1} - \frac{1}{4} (\delta_{i1} - \delta_{iN}) \left(TH_{1N} \otimes (1 - T^2) + (1 - T^{-2}) \otimes T^{-1} H_{1N} \right), \\
& & i \in \{1, 2, \dots, N\}, \\
\Delta(E_{i,N+1}) &= E_{i,N+1} \otimes T^{-(\delta_{i1} + \delta_{iN})/2} + T^{(\delta_{i1} - \delta_{iN})/2} \otimes E_{ji} + \frac{\hbar}{4} T^{-1} \delta_{iN} E_{1N} \otimes \left(T^{-1/2} H_{1N} + \right. \\
& & \left. H_{1N} T^{-1/2} \right) - \frac{\hbar}{4} \left(T^{1/2} H_{1N} + H_{1N} T^{1/2} \right) \otimes \delta_{iN} T E_{1N} \quad i \in \{1, 2, \dots, N\}, \\
\Delta(E_{N+1,i}) &= E_{N+1,i} \otimes T^{(\delta_{i1} - \delta_{iN})/2} + T^{-(\delta_{i1} + \delta_{iN})/2} \otimes E_{N+1,i} - \frac{\hbar}{4} T^{-1} \delta_{i1} E_{N+1,N} \otimes \left(T^{-1/2} H_{1N} + \right. \\
& & \left. H_{1N} T^{-1/2} \right) + \frac{\hbar}{4} \left(T^{1/2} H_{1N} + H_{1N} T^{1/2} \right) \otimes \delta_{i1} T E_{N+1,N} \quad i \in \{1, 2, \dots, N\}.
\end{aligned} \tag{4.2}$$

The commutator rules of $\mathcal{U}_\hbar(sl(N|1))$ can be evaluated by direct calculations.

Paralleling the earlier cases, the universal \mathcal{R}_\hbar -matrix of $\mathcal{U}_\hbar(sl(N|1))$ is given by

$$\mathcal{R}_\hbar = \exp\left(-\hbar E_{1N} \otimes TH_{1N}\right) \exp\left(\hbar TH_{1N} \otimes E_{1N}\right), \tag{4.3}$$

where $E_{1N} = \hbar^{-1} \ln T$. This element can be connected to the results obtained by the contraction process by a suitable twist operator that can be derived as a series expansion in \hbar .

5 Conclusion

In general, a class of nonlinear maps exists relating the Jordanian quantum (super)algebras and their classical analogues. Here we have used a particular map realizing Jordanian $\mathcal{U}_\hbar(sl(N|1))$ superalgebra for an arbitrary N . This map arises naturally from our contraction process defined in (1.1). Let us just recall that the more important advantages of our procedure are:

- The algebraic commutation relations are deformed.
- The coalgebraic structure is simpler.
- The map obtained, by our contraction process, permits immediate explicit construction of the finite-dimensional irreps.

- The Ohn's $\mathcal{U}_h(sl(2))$ algebra is embedded as a Hopf subalgebra in our construction. Therefore, the Jordanian $\mathcal{U}_h(sl(N|1))$ superalgebra arising from our method corresponds to the classical r -matrix $r = h_{1N} \otimes e_{1N} - e_{1N} \otimes h_{1N}$.

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